Positional Tolerance And Relative Accuracy

2005 Career Advancement Program
Land Surveyors Association of Washington
North Puget Sound Chapter

Texts:
Surveying Measurements and their Analysis
R.B. (Ben) Buckner, Ph.D, PE, PLS 1983

Surveying – Theory and Practice
Davis, Foote, Anderson, Mikhail 1981

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A land surveyor is, by state law, qualified to make measurements of the land or below water. Those measurements, together with the knowledge base of interpreting maps and deeds, lead to the establishment of laying out deed lines between property owners. By licensure, he is qualified to provide expert testimony regarding measurements in courts of law. Therefore, he must be in control of those measurements.

There are errors in every measurement made by a person or made by an instrument manufactured by a person. Common standards of practice afford the surveyor basic skills in measurement. By analyzing measurements and acknowledging the presence of error, the surveyor will become more confident in his ability to exercise his professional judgment and will be able to qualify every measurement made to a reasonable degree of accuracy and precision.

Some Basic Definitions:

- Land Surveying – The science and art of making such measurements as are necessary to determine the relative position of points above, on, or beneath the surface of the earth, or to establish such points in a specified position.
- Measurement – The act or process of ascertaining the extent, dimensions, or quantity of something. An estimate of a quantity. Anything measured is inexact.
- Count – To check over one by one to determine the total number. An exact quantity of a sample or population.
- Blunder – A careless mistake.
- Error – A deviation from accuracy or correctness. The difference between an observed or computed value and the true value.

- Plane Surveying – Surveying under the assumption that the earth is a plane surface and that all north–south lines are parallel.
- Geodetic Surveying – Surveying considering the earth's curvature and convergence of meridians.

Units of Measurement:

Linear Units – Predetermined lengths in one dimension to determine distances.
(Inches, Feet, Yard, Rod, Chain, Mile, Meter.)

Angular Units – Predetermined portions of a circle to determine angles.
(Seconds, Minutes, Degrees, Radians, Grads.)

Area Units – Predetermined distances in two dimensions to determine area.
(Acre, Hectare.)

Types of Errors:

- Systematic Errors – Errors conforming to known mathematical and physical laws. They are predictable and remain the same under set conditions but may vary in magnitude.
- Random Errors – An error whose presence is unavoidable and unpredictable but generally behaves according to mathematical laws and tends to cancel but never completely do.

Precision and Accuracy:

- Precision – The agreement among readings of the same quantity. Precision relates to the refinement in manufacture of equipment and the care and refinement in making measurements. If precision in measurement is high, the random errors should be small. Precision relates to the method of measurement.
- Accuracy – The agreement of the measurement or measurements with the true value. A value that is closer to the true value is more accurate than one which is farther than the true. Accuracy relates to the result.

Sources of Errors in Land Surveying:

- Natural Errors – Caused by effects from nature, including temperature, humidity, gravity, atmospheric pressure, atmospheric refraction, curvature of the earth, wind, tension, etc.
- Instrumental Errors – Caused by either the initial manufacture of a measuring instrument or by wear and//or maladjustment of the measuring instrument. Initial manufacturing errors are predictable and are, therefore, systematic. Wear and/or maladjustment errors are unpredictable and are, therefore, random.
- Personal Errors – Caused by the inability of humans to perceive anything exact, including readings, aligning cross-hairs and other marks or centering devices. Aside from this innate inability which affects all of us, people vary in their manual dexterity, experience, training, intelligence, motivation and the desire to employ appropriate care.

Direct and Indirect Measurements:

- Direct Measurements – Measuring directly between points or lines.

- Indirect Measurements – Measurement computed from other measurements.

Significant Figures:

- Or Significant Digits – The number of digits that are meaningful in a measurement or quantity.
  a. Zeroes used merely to indicate the position of the decimal point are not significant. For example, the number 0.00618 has three significant figures.
  b. If zeroes are recorded at the end of a measurement, they are significant. For example, the number 61.410 has five significant figures. The number 0.0060 has two significant figures.
  c. Zeroes between non-zero digits are significant. For example, the number 12.1003 has six significant figures.
  d. In a number ending with one or more zeroes to the left of the decimal, a special indication of the exact number of significant figures must be made. For example, the number 615,000 has six significant figures. The number 615,000 has three significant figures. The number 360 could have either two or three significant figures.
  e. Truncating a number means to delete all digits to the right of the number of significant figures. For example, the number 3.141592654 truncated to five significant figures is 3.1415.
  f. Rounding a number means to either change the last number of significant figure up one if the number is greater than 5 or down one if the number is less than or equal to 5. For example, the number 3.141592654 rounded to five significant figures is 3.1416.
  g. The precision of a number, lacking other methods of determining precision, is generally considered to be plus or minus ½ of the last place. For example, the number 42.81 is understood to be 42.81 plus or minus 0.005 or an uncertainty of plus or minus 0.005.
  h. If a number has an uncertainty value, one can discover how many significant figures are contained in a measured quantity. If the number 42.81 has an uncertainty value of plus or minus 0.5, then the number should end in the units' column or it should be stated as 43.

- Adding or Subtracting – The number of significant figures in the sum or difference is determined by the fewest decimal places in the numbers added or subtracted. For example 14.623 + 12.01 + 1.0 = 27.633 but since one of the numbers in the sum has only one place to the right of the decimal, the sum can only be significant to one place to the right of the decimal or 27.6.
- Multiplying or Dividing – The number of significant figures in the product or quotient is determined by the fewest number of significant figures in the values used. For example, 14.29 x 0.051 = 0.73, because 0.051 has two significant figures and 14.29 has four significant figures, the product may only have two.
- Conversion factors or constants are not measured quantities, and thus do not determine significant figures. For example, 1342.5 inches / 12 inches/foot + 111.87

feet (five significant figures in the number of inches yields five significant figures in the quotient.

- To avoid round-off errors when using conversion factors or constants that contain a large or infinite number of digits (or are not counts) use one extra figure in such values. For instance, a trigonometric number or the constant Pi. For example, using Pi, multiply 165.41 by Pi. It would be incorrect to multiply 165.41 x 3.14 = 519.39. It would be correct to multiply 165.41 x 3.14159 = 519.65 since the product has no round-off error.
- To avoid round-off errors when using computed values in subsequent calculations, carry one extra figure throughout the intermediate calculations. For example, 131.46 x 32.68 = 4296 as a final answer, but use 4296.1 in any further calculations.
- Round off final answers to the significant figures warranted by the measurements and the rules on computing as cited above.

Errors in Angular Measurements:

a. Reading Error – The inability to read a vernier or circle to some precision.
b. Pointing Error – The inability to place the cross-hairs in the telescope exactly centered on a target.
c. Instrument Centering Error – The inability to place a theodolite or transit directly over a point.
d. Target Centering Error – The inability to place a target directly over a point.
e. Instrument Leveling Error – The inability to perfectly level a theodolite or transit.

Errors in Distance Measurements:

- Generally are associated with the ability to read a measuring tape or a manufacturer's standard error.

Propagation of Random Errors:

- Error in a Sum – Where each error is different, for example, in an angular analysis. The total error is the square root of the sum of the squares.
- Error in a Series – Where each error is the same, for example, in a taped distance of multiple lengths of tape. The total error is the error multiplied by the square root of the number of times the error occurs.
- Error in a Product – Similar to an error in a sum, for example, given uncertainty in the measurements on the sides of a rectangle, the error in a product will yield the uncertainty associated in the computed area.

Statistics:

- Statistics is that branch of the mathematical sciences that analyses a random sample and provides conclusions for an entire population.
- Statistical theory relies on several assumptions:
  1. The random sample that is analyzed is random.
  2. The random sample is sufficiently large to provide accurate data for theoretical conclusions.

3. The random sample is a true representation of the entire population.
4. Any errors in the random sample are random.

Definitions of Statistical Terms:

- Sample Size – The number of observations or measurements in a sample.
- Mean – Or arithmetic mean, the sum of the observations or measurements of a sample divided by the sample size.
- Median – The middle value of a sample.
- Mode – The value which occurs most frequently in a sample.
- Residual – The arithmetic difference between an individual value in a sample and the mean of that sample.
- Probability Curve – A plot of points resembling the shape of a bell centered on the mean value of a sample. Generally, the Y-axis of the plot is where the mean value is plotted.
- Points of Inflexion – The points on the probability curve where the curve changes shape from concave down to concave up. Given that the area under the probability curve and above the X-axis is 100% of the total area, the area under the probability curve between two lines drawn perpendicular to the X-axis through the points of inflexion equals 68.3% of the total area.
- Standard Deviation – Or Standard Deviation of a Single Value, the number representing the distance on the X-axis from the mean to the points of inflexion. Theoretically, the standard deviation is symmetric about the mean value. It is equal to the square root of the sum of the squared residuals divided by the sample size minus one. Generally, it is represented by the Greek letter, sigma.
- Standard Error of the Mean – An uncertainty statement regarding the average (mean value) of a set rather than a randomly selected single value, as the standard deviation is. The uncertainty is with respect to the true value, not the mean value. It is equal to the standard deviation divided by the square root of the sample size. The value could be positive or negative.
- Level of Certainty – Or Percent Probability, it represents a level or degree of confidence regarding an error or uncertainty statement. One multiplied by the standard deviation, or one-sigma, is a 68.3% level of certainty. Two multiplied by the standard deviation, or two-sigma, is a 95% level of certainty. Three multiplied by the standard deviation, or three-sigma, is a 99.7% level of certainty.

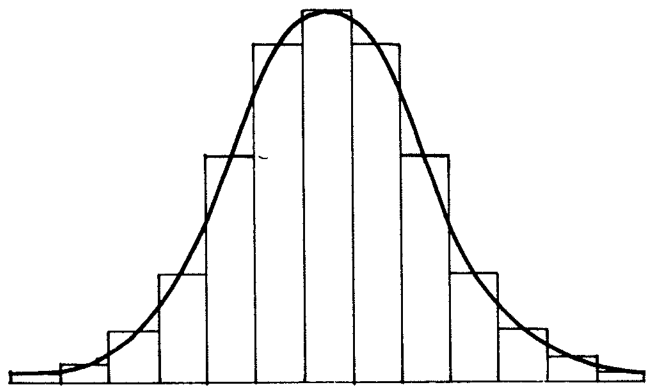


Figure 4.5 Histogram and Probability Curve

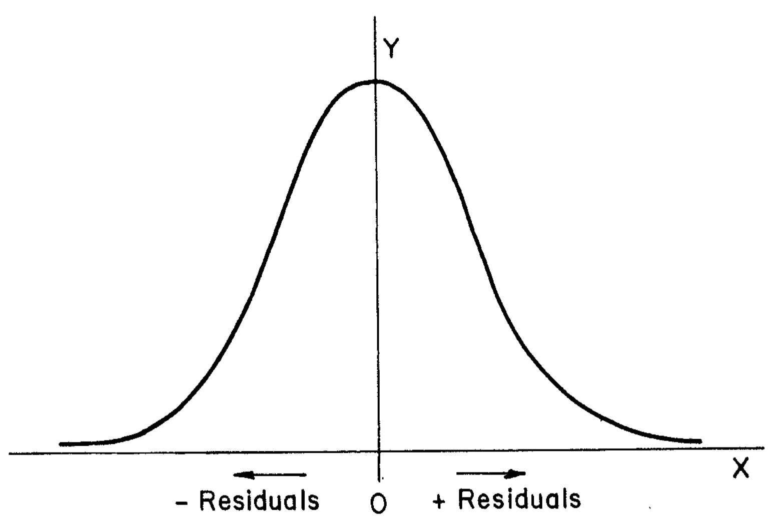


Figure 4.6 Normal Probability Curve

TABLE 4.5

CALCULATION OF STANDARD DEVIATION

SET 1

Wild T - 2 (Coincidence Circle)

Observation	Reading	ν	ν^2
1	31.8	+0.6	0.36
2	32.3	+1.1	1.21
3	31.9	+0.7	0.49
4	30.2	−1.0	1.00
5	31.4	+0.2	0.04
6	32.7	+1.5	2.25
7	31.1	−0.1	0.01
8	31.0	−0.2	0.04
9	31.2	0.0	0
10	30.3	−0.9	0.81
11	30.4	−0.8	0.64
12	30.9	−0.3	0.09
13	31.4	+0.2	0.04
14	31.0	−0.2	0.04
15	31.4	+0.2	0.04
16	30.8	−0.4	0.16
17	31.8	+0.6	0.36
18	31.0	−0.2	0.04
19	30.6	−0.6	0.36
20	30.4	−0.8	0.64
21	31.4	+0.2	0.04
22	30.2	−1.0	1.00
23	31.0	−0.2	0.04
24	31.5	+0.3	0.09
25	31.7	+0.5	0.25
	Σx_i = 779.4		$\Sigma \nu^2$ = 10.04

$$\bar{x} = \frac{779.4}{25} = 31.2$$

$$\sigma = \sqrt{\frac{\Sigma \nu^2}{n-1}} = \sqrt{\frac{10.04}{24}} = \sqrt{0.418} = \pm\, 0.65$$

Note: All readings are in seconds.

TABLE 4.6

LEVELS OF CERTAINTY

Name of Error	Symbol	Value	% Certainty
Probable	E_{50}	0.6745σ	50
Standard deviation	σ	1σ	68.3
90% Error	E_{90}	$1.6449\ \sigma$	90
Two-Sigma or 95% Error	E_{95}	2σ	95
99% Error	E_{99}	$2.5\ \sigma$	99
Three-Sigma	$E_{99.7}$	3σ	99.7

(Note: Acutally, the 95% error is closer to 1.96σ, but 2σ is often accepted as a convenient conversion. The conversions for the 50% and 90% errors are normally not needed to the significant figures given in the table.)

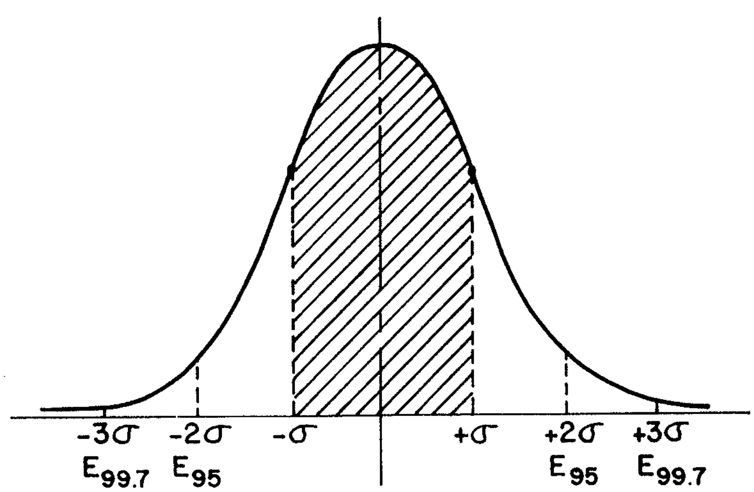


Figure 4.7 Theoretical Precision Indexes

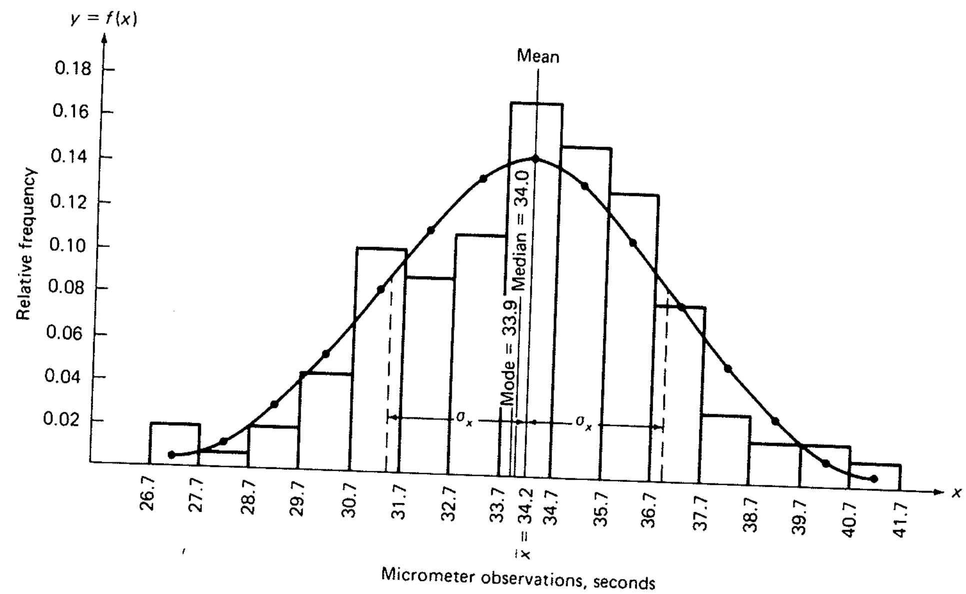


Fig. 2.6 Histogram and density distribution curve for micrometer readings.

2.18. Accuracy and precision The term *accuracy* refers to the closeness between measurements and their expectations (or, in conventional terms, to their true values). The farther a measurement is from its expected value, the less accurate it is. *Precision*, on the other hand, pertains to the closeness to one another of a set of repeated observations of a random variable. Thus, if such observations are closely clustered together, then the observations are said to have been obtained with high precision. It should be apparent, then, that observations may be precise but not accurate if they are closely grouped together but about a value that is different from the expectation (or true value) by a significant amount. Also, observations may be accurate but not precise if they are well distributed about the expected value but are significantly disbursed from one another. Finally, observations will be both precise and accurate if they are closely grouped around the expected value (or the distribution mean).

One of the examples used most often to demonstrate the difference between the two concepts of accuracy and precision is that of rifle shot groupings. Figure 2.7 shows three different types of groupings that it is possible to obtain. From the discussion above, group (*a*) is both accurate and precise, group (*b*) is precise but not accurate, and group (*c*) is accurate but not precise. One of the harder notions to accept is that case (*c*) is in fact *accurate*, even though the scatter between the different shots is rather large. A justification which may help is that we can visualize that the center of mass (which is equivalent to the expected value of the different shots) turns out to be very close to the target center (which is the true value).

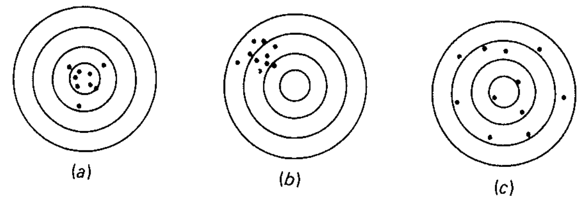


Fig. 2.7 Rifle shot groupings.

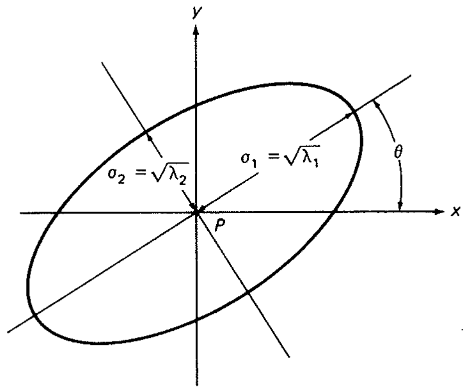


Fig. 2.8 Error ellipse.

2.19. Error ellipses The variance or standard deviation are measures of precision for the one-dimensional case of an angle or a distance, for example. In the case of two-dimensional problems, such as the horizontal position of a point, error ellipses may be established around the point to designate precision regions of different probabilities. The orientation of the ellipse relative to the x,y axes system (Fig. 2.8) depends on the correlation between x and y. If they are uncorrelated, the ellipse axes will be parallel to x and y. If the two coordinates are of equal precision, or $\sigma_x = \sigma_y$, the ellipse becomes a circle.

Considering the general case where the covariance matrix for the position of point P is given as

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \tag{2.39}$$

The semimajor and semiminor axes of the corresponding ellipse are computed in the following manner. First, a second-degree polynomial (called the characteristic polynomial) is set up using the elements of Σ as

$$\lambda^2 - (\sigma_x^2 + \sigma_y^2)\lambda + (\sigma_x^2\sigma_y^2 - \sigma_{xy}^2) = 0 \tag{2.40}$$

The two roots λ_1, λ_2 of Eq. (2.40) (which are called the eigenvalues of Σ) are computed and their square roots are the semimajor and semiminor axes of the *standard error ellipse*, as shown in Fig. 2.8. The orientation of the ellipse is determined by computing θ between the x axis and the semimajor axis from

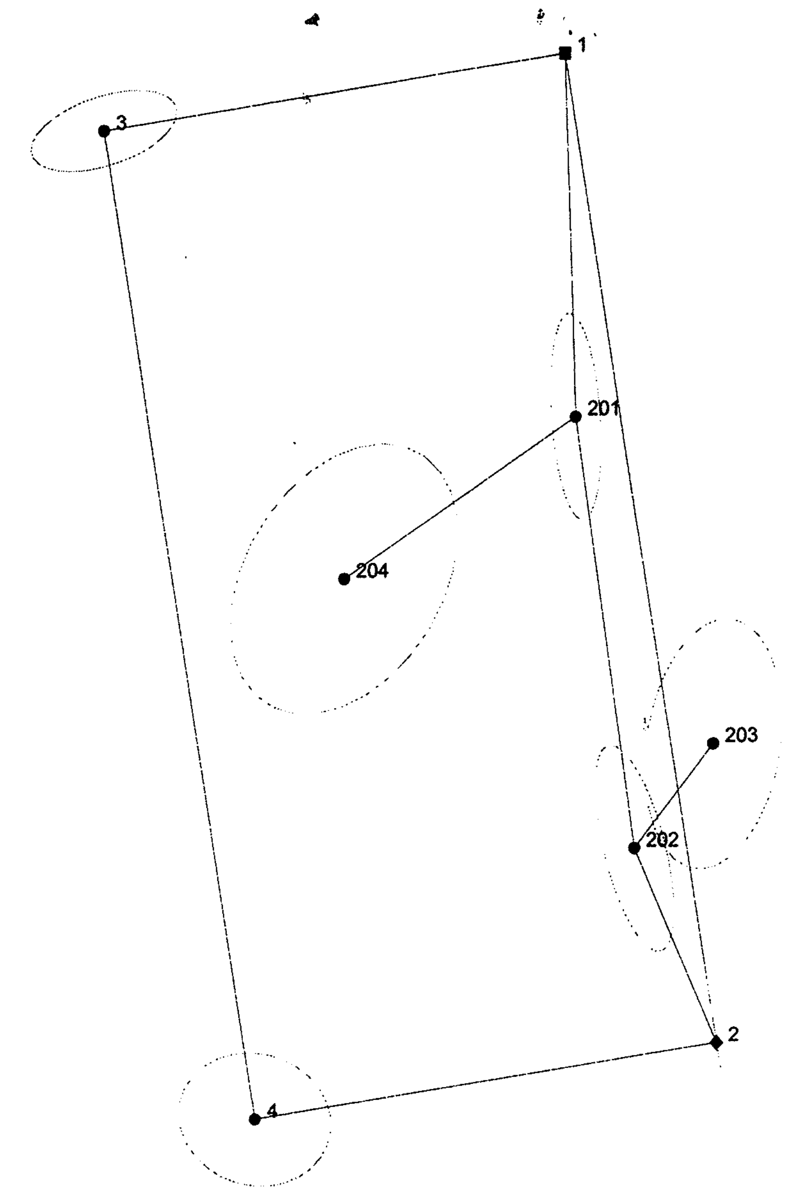
$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2} \tag{2.41}$$

The quadrant of 2θ is determined from the fact that the sign of $\sin 2\theta$ is the same as the sign of σ_{xy}, and $\cos 2\theta$ has the same sign as $(\sigma_x^2 - \sigma_y^2)$. Whereas in the one-dimensional case, the probability of falling within $+\sigma$ and $-\sigma$ is 0.6827 [see Eq. (2.14)], the probability of falling on or inside the standard error ellipse is 0.3935. In a manner similar to constructing intervals with given probabilities as in Eq. (2.15) for the one-dimensional case, different-size ellipses may be established, each with a given probability. It should be obvious that the larger the size of the error ellipse, the larger is the probability. Using the standard ellipse as a base, Table 2.4 gives the scale multiplier k to enlarge the ellipse and the corresponding probability (see Mikhail, Ref. 8).

Table 2.4

k	1.000	1.177	2.146	2.447	3.035
p	0.394	0.500	0.900	0.950	0.990

ERROR ELLIPSES IN A TRAVERSE



History of the US Survey Foot

Survey distance measurements have been made throughout history using a variety of units. Ancient units, for example, included the cubit, palm and digit. Units such as the *rod* were employed in Medieval England and later in the United States and other countries. The rod has also been called perch, rood, pole and other names. Even later, the *chain* was defined as a unit of measurement, following the invention of a Gunter's chain by Edmund Gunter in 1653. Rods and chains were used for early land surveys and such units are still seen in old land surveys and descriptions.

The English foot is still employed in surveying, particularly in engineering measurements. In order to facilitate computations, surveyors divide the foot into decimal units (tenths, hundredths, etc.) rather than inches and fractions of inches.

In 1791, the meter was introduced as the unit of linear measurement in the metric system, being defined as 10^{-7} of the earth's quadrant. The definition was changed, in practice, in 1799 when it was defined by the length of a prototype meter bar.

All measuring devices such as wires, tapes and rods may be calibrated in terms of the meter, so that all distances can be measured uniformly in the SI (metric) system. In 1866, the U.S. Congress legalized the use of the metric system and adopted the Paris International prototype meter. The meter was then equivalent to 39.37 inches exactly, which yields:

1	Foot	=	0.3048006096 meters,
1	Meter	=	3.28083333 feet,
1	Inch	=	2.540005080 centimeters.

The meter was redefined again in 1960 by the 11th General Conference of Weights and Measures in terms of the wavelength of the radiation produced by a specified quantum transition of the Krypton-86 isotope. This permits a more accurate reproduction of the meter than is possible through use of a bar. However, through this new definition, the result was:

1	Foot	=	0.3048 meters exactly,
1	Meter	=	3.280839895 feet,
1	Inch	=	2.54 centimeters exactly.

However, the surveying profession in the United States continued to use the "old" (prior to 1960) conversion because the national geodetic network of horizontal control is based on this.

In 100 miles, using the "U.S. Survey Foot" (the 1866 definition), there are 160,934.7219 meters.
In 100 miles, using the "International Foot" (the 1960 definition), there are 160,934.4000 meters.